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ON THE TYCHONOFF FUNCTOR AND w-COMPACTNESS

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The main purpose of this paper is to settle the following problem concerning a product formula for the Tychonoff functor τ, by introducing the notion of w-compact spaces: Characterize a topological space X such that $\tau(X\times Y)=\tau(X)\times\tau(Y)$ for any topological space Y. We also study the properties of w-compact spaces, and it is proved that, for any family $\{X_\alpha\}$ of w-compact spaces, the product $\prod X_\alpha$ is also w-compact and $\tau(\prod X_\alpha)=\prod\tau(X_\alpha)$.

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1. Introduction

In this paper we mean by a space a topological space with no separation axiom unless otherwise specified, and we denote by N and I the set of natural numbers and the closed unit interval respectively.

For a space X, let I^X be the set of all continuous maps $\varphi : X\to I$ and consider a continuous map $\Phi_X : X\to P(X)$ from X to the product space $P(X)=\prod\{I_\varphi \mid \varphi\in I^X\}$ defined by $\Phi_X(x)=(\varphi(x))\in P(X)$, where $I_\varphi=I$ for any $\varphi\in I^X$. Let us put $\tau(X)=\Phi_X(X)\subset P(X)$. Then for a continuous map $f:X\to Y$ we have a continuous map $\tau(f):\tau(X)\to\tau(Y)$ by defining $\tau(f)(t)$ to be the point of $\tau(Y)$ whose ψ-coordinate is the $\psi\circ f$-coordinate of $t\in\tau(X)$, where $\psi\in I^Y$, and the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Phi_X"'] & Y \arrow[d, "\Phi_Y"] \\ \tau(X) \arrow[r, "\tau(f)"'] & \tau(Y) \end{tikzcd}$$

is commutative. Thus τ is a covariant functor from the category of topological spaces and continuous maps into itself, which we call the Tychonoff functor after K. Morita [4]. The Tychonoff functor is the reflector from the category above to the full subcategory of Tychonoff spaces. Hence the following statements are valid:

(I) For any cozero-set G of X, $\Phi_X(G)$ is a cozero-set of $\tau(X)$ with $\Phi_X^{-1}\Phi_X(G)=G$.

(II) For any family of spaces $\{X_\alpha \mid \alpha \in \Omega\}$ there exists a continuous map $f:\tau(\prod X_\alpha)\to\prod\tau(X_\alpha)$ such that

$$\prod \Phi_{X_\alpha} = f\circ\Phi_{\prod X_\alpha}.$$

If f is homeomorphic, we put

$$\tau(\prod X_\alpha)=\prod\tau(X_\alpha).$$

Concerning this equality for the case of any two spaces X and Y, the following theorem was proved by R. Puppier [6].

Theorem 1.1. *If X is a locally compact Hausdorff space, then the equality $\tau(X\times Y)=\tau(X)\times\tau(Y)$ holds for any space Y.*

Recently K. Morita has given another proof of this theorem by proving the following result.

Theorem 1.2. *For any spaces X and Y, $\tau(X\times Y)=\tau(X)\times\tau(Y)$ if and only if for any cozero-set G of $X\times Y$ and for any point $(x,y)\in G$ there exists a rectangular cozero-set $U_x\times V_y$ of $X\times Y$ such that $(x,y)\in U_x\times V_y\subset G$.*

If U and V are cozero-sets of X and Y respectively, then $U\times V$ is called a rectangular cozero-set of $X\times Y$.

Furthermore, with the aid of Theorem 1.2, S. Oka [5] has proved the following result.

Theorem 1.3. *Let X be a Tychonoff space. Then the following conditions are equivalent.*

(1) *X is locally compact.*

(2) *$\tau(X\times Y)=X\times\tau(Y)$ for any space Y.*

However the following problem still remains open: Characterize a space X such that the equality $\tau(X\times Y)=\tau(X)\times\tau(Y)$ holds for any space Y.

The main purpose of this paper is to give the solution for this problem by introducing the notion of w-compact spaces and to study the remarkable properties of w-compact spaces.

A subset P of a space X is called a τ-open set of X if it is a union of cozero-sets of X. Obviously a subset P of a space X is τ-open if and only if $P=\Phi_X^{-1}(Q)$ for some open set Q of $\tau(X)$. Furthermore the union and the finite intersection of τ-open sets are also τ-open.

Definition 1.4. A space X is *w-compact* if for any family $\{P_\lambda\}$ of τ-open sets of X with the finite intersection property, $\bigcap \bar{P}_\lambda \neq \emptyset$.

As is easily seen, this definition is equivalent to the following: A space X is w-compact if for any collection $\{A_\alpha\}$ of closed sets of X such that it is closed under the finite intersection and each A_α contains a non-empty cozero-set of X, we have $\bigcap A_\alpha \neq \emptyset$.

Every compact space is w-compact, but the converse is not true in general. In fact, it is known that there exists a regular T_1 space X, containing at least two points, in which every continuous map from X to the real line R is constant. Such a space X is w-compact but not compact. The basic properties of w-compact spaces will be mentioned in Section 2.

Now, by making use of the notion of w-compact spaces, we can solve the problem above.

Theorem 1.5. *For a space X, the following conditions are equivalent.*

(1) *For each point x of X, there exists a cozero-set nbd (=neighborhood) W of x such that $\bar{W}$ is w-compact.*

(2) $\tau(X\times Y)=\tau(X)\times\tau(Y)$ *for any space Y.*

(3) $\tau(X\times Y)=\tau(X)\times\tau(Y)$ *for any regular T_1 space Y.*

As is shown in Section 2, there exists a w-compact Hausdorff space being not locally compact, and hence Theorem 1.5 shows that, in case X is not Tychonoff, (1) and (2) in Theorem 1.3 are not equivalent.

In case Y is any k-space, the following result was proved by R. Puppier [6], where we mean by a k-space a Hausdorff k-space.

Theorem 1.6. *If $\tau(X)$ is a locally compact space, then* $\tau(X\times Y)=\tau(X)\times\tau(Y)$ *for any k-space Y.*

However, more precisely, we can prove the following theorem.

Theorem 1.7. *For a space X, the following conditions are equivalent.*

(1) *For each point x of X and for any cozero-set nbd U of x, there exists a cozero-set W of X such that $x\in W\subset U$ and $\bar{W}$ is pseudocompact (that is, every continuous map $f:\bar{W}\to R$ is bounded).*

(2) $\tau(X\times Y)=\tau(X)\times\tau(Y)$ *for any k-space Y.*

(3) $\tau(X\times Y)=\tau(X)\times\tau(Y)$ *for any regular k-space Y.*

Finally, as for the product of w-compact spaces, we shall establish the following theorem.

Theorem 1.8. *Let $\{X_\alpha\}$ be any family of w-compact spaces. Then the following statements are valid.*

(1) *The product* $\prod X_\alpha$ *is also w-compact.*
(2) *The equality* $\tau(\prod X_\alpha) = \prod \tau(X_\alpha)$ *holds.*

The first of the theorem is a generalization of a famous theorem of Tychonoff concerning the product of compact spaces and the second is an infinite product formula for the Tychonoff functor which is proved with the aid of the first.

As the applications of Theorem 1.8 (2), we can obtain the following corollaries.

Corollary 1.9. *Let* $\{X_\alpha \mid \alpha \in \Omega\}$ *be any family of w-compact spaces. Then for any continuous map* $f:\prod X_\alpha \to I$, *there exists a countable subset* Ω' *of* Ω *and a continuous map* $g:\prod\{X_\alpha \mid \alpha \in \Omega'\} \to I$ *such that* $f = g \circ \pi$, *where* $\pi:\prod X_\alpha \to \prod\{X_\alpha \mid \alpha \in \Omega'\}$ *is the projection.*

This is a generalization of Y. Mibu's theorem [3] for the case of compact Hausdorff spaces.

Corollary 1.10. *Let* $\{X_\alpha \mid \alpha \in \Omega\}$ *be any family of w-compact spaces, and let* Y *be a Tychonoff space with the* G_δ*-diagonal. Then for any continuous map* $f:\Sigma(a) \to Y$, *there exists a countable subset* Ω' *of* Ω *and a continuous map* $g:\prod\{X_\alpha \mid \alpha \in \Omega'\} \to Y$ *such that* $f = g \circ \pi$, *where* $\pi:\prod X_\alpha \to \prod\{X_\alpha \mid \alpha \in \Omega'\}$ *is the projection.*

This is a generalization of R. Engelking's theorem [1] for the case of compact T_1 spaces, and $\Sigma(a)$ denotes the Σ-product of spaces $\{X_\alpha \mid \alpha \in \Omega\}$ for a fixed point $a = (a_\alpha) \in \prod X_\alpha$, i.e.,

$$\Sigma(a) = \{(x_\alpha) \in \prod X_\alpha \mid \operatorname{card}\{\alpha \mid x_\alpha \neq a_\alpha\} \leq \aleph_0\}.$$

Theorems 1.5 and 1.7 mentioned above were announced at the Colloquium on Topology in Budapest, 1978.

2. Properties of w-compact spaces

In this section we shall study basic properties of w-compact spaces.

Proposition 2.1. *If* X *is a w-compact space, then* $\tau(X)$ *is compact.*

Proof. We first notice that $\tau(X)$ is compact if and only if any family of zero-sets of X with the finite intersection property has the non-empty intersection. Now suppose that $\tau(X)$ is not compact. Then there exists a cozero-set cover $\mathcal{U}$ of X having no finite subcover, and hence we can take a zero-set refinement $\mathcal{Z} = \{Z_\gamma \mid \gamma \in \Gamma\}$ of $\mathcal{U}$ and a cozero-set refinement $\{H_\gamma \mid \gamma \in \Gamma\}$ of $\mathcal{Z}$ such that $H_\gamma \subset Z_\gamma$ for each $\gamma \in \Gamma$. Let us put

$$A_\gamma = X - H_\gamma, \qquad B_\gamma = X - Z_\gamma$$

for any $\gamma \in \Gamma$. Then $\{B_\gamma \mid \gamma \in \Gamma\}$ is a family of cozero-sets of X with the finite intersection property and $\bigcap \bar{B}_\gamma \subset \bigcap A_\gamma = \emptyset$. Therefore X is not w-compact, which is a contradiction. This completes the proof.

Corollary 2.2. *Let X be a Tychonoff space. If X is w-compact, then it is compact.*

The following proposition shows that the converse of Proposition 2.1 is not true in general.

Proposition 2.3. *There exists a non-w-compact, regular Hausdorff space X such that $\tau(X)$ is compact.*

Proof. Let ω_1 be the first uncountable ordinal and let us put

$$S = W(\omega_1 + 1) \times W(\omega_1 + 1) - (\omega_1, \omega_1),$$

where $W(\omega_1 + 1)$ is the set of all ordinals $\alpha \leq \omega_1$ with the usual interval topology. We put further

$$P = \{(\alpha, \omega_1) \mid \alpha < \omega_1\}, \qquad Q = \{(\omega_1, \beta) \mid \beta < \omega_1\}$$

in S. For each n, let S_n be the copy of S and φ_n a homeomorphism of S onto S_n. In the topological sum $\bigcup S_n$ of $\{S_n \mid n \in N\}$, we identify a point $\varphi_{2m-1}(p)$ with $\varphi_{2m}(p)$ for $p \in P$ and a point $\varphi_{2m}(q)$ with $\varphi_{2m+1}(q)$ for $q \in Q$. By this identification we have a quotient space Y, which is locally compact Hausdorff. Now let X be a space obtained by adding a new point ξ to Y and introducing the topology in X as follows: the base at ξ is given by the totality of the sets $(Y - \bigcup_{i=1}^{n} \varphi_i(S)) \cup \{\xi\}$, $n \in N$, and the base at $x \neq \xi$ is the same as in Y. Then X has the following properties:

(1) X is regular Hausdorff but not Tychonoff.
(2) $\tau(X)$ is compact.
(3) X is not w-compact.

Indeed, (1) and (2) follow from the fact that any cozero-set of X containing ξ has to contain a set of the form

$$\{\xi\} \cup \bigcap_{i=1}^{n} (X - \varphi_i(S)) \cup \bigcup_{i=1}^{n} \varphi_i(T_\alpha)$$

for some $\alpha < \omega_1$, where $T_\alpha = \{(\lambda, \mu) \mid \lambda, \mu > \alpha\} \subset S$, and (3) follows from the fact that $\{\varphi_1(T_\alpha) \mid \alpha < \omega_1\}$ is a family of closed sets of X such that $\varphi_1(T_\alpha)$ contains isolated points of X and

$$\bigcap \{\varphi_1(T_\alpha) \mid \alpha < \omega_1\} = \emptyset.$$

Thus the proof is completed.

The closed subset of a w-compact space is not always w-compact. For example, let X be a space obtained by introducing the following topology in the closed unit interval I: the base at $x \neq 0$ is given by the family $\{U_\varepsilon(x) \mid \varepsilon > 0\}$ of ε-nbds of x and the base at $x = 0$ is given by the family $\{U_\varepsilon(0) - A \mid \varepsilon > 0\}$, where $A = \{1/n \mid n \in N\}$. Then it is easily shown that X is w-compact but the closed subset A of X is not w-compact. We notice that X is completely Hausdorff (that is, for any different points a and b of X there is a continuous map $f: X \to R$ such that $f(a) \neq f(b)$) but is not regular. Hence this example also shows that there exists a completely Hausdorff, w-compact space being not locally compact.

Proposition 2.4. *If U is a cozero-set of a space X such that $\bar{U}$ is w-compact, then $\overline{U \cap H}$ is w-compact for any cozero-set H of X.*

Proof. We may assume that $U \cap H$ is not empty. Let $\{A_\alpha\}$ be a family of closed sets of $\overline{U \cap H}$ closed under the finite intersection and satisfying the condition that for each α, A_α contains a non-empty cozero-set G_α of $\overline{U \cap H}$. Since $G_\alpha \cap (U \cap H)$ is a non-empty cozero-set of X for each α, $\{A_\alpha\}$ is considered as a family of closed sets of $\bar{U}$ with the same properties as in $\overline{U \cap H}$. Hence we have $\bigcap A_\alpha \neq \emptyset$. This completes the proof.

The following proposition is obvious.

Proposition 2.5. *If $f: X \to Y$ is a continuous map from a w-compact space X onto a space Y, then Y is also w-compact.*

Proposition 2.6. *Let U be a cozero-set of a space X such that $\bar{U}$ is not w-compact. Then there exists a space Y with only one non-isolated point y_0 of Y and a continuous map $h: X \times Y \to I$ such that*

$$h(z) = 1 \quad \textit{for } z \in (X \times y_0) \cup ((X - U) \times Y)$$

and

$$h^{-1}(0) \cap (U \times y) \neq \emptyset \quad \textit{for } y \in Y - y_0.$$

Proof. Since $\bar{U}$ is not w-compact, there is a family $\{A_\alpha \mid \alpha \in \Omega\}$ of closed sets of $\bar{U}$ with $\bigcap A_\alpha = \emptyset$ such that it is closed under the finite intersection and that each A_α contains a non-empty cozero-set G_α of $\bar{U}$. If we put $H_\alpha = G_\alpha \cap U$ for each $\alpha \in \Omega$, then H_α is a non-empty cozero-set of X. We now introduce the order in Ω in such a way that $\alpha \leq \beta$ if $A_\alpha \supset A_\beta$, and construct a space $Y = \Omega \cup \{\xi\}$, where ξ is a new point, with the topology as follows: each point of Ω is open and the totality of the sets $U_\beta = \{\gamma \in \Omega \mid \beta \leq \gamma\} \cup \{\xi\}$, $\beta \in \Omega$, is a base at ξ. Since for each $\alpha \in \Omega$ there is a non-empty zero-set L_α of X contained in H_α, there exists a continuous map $h_\alpha: X \times \alpha \to I$ such that

$$h_\alpha(z) = 1 \quad \text{for } z \in (X - H_\alpha) \times \alpha,$$

$$h_\alpha(z) = 0 \quad \text{for } z \in L_\alpha \times \alpha.$$

Finally we define a map $h: X \times Y \to I$ as follows:

$$h(z) = h_\alpha(z) \quad \text{for } z \in X \times \alpha,$$

$$h(z) = 1 \qquad \text{for } z \in X \times \xi.$$

Then it is easily seen that h is a continuous map satisfying the required properties. Thus the proof is completed.

We notice that the above mentioned space Y is paracompact Hausdorff.

Proposition 2.7. *For a space X, the following properties are equivalent.*

(1) *X is w-compact.*

(2) *The projection $\pi_Y: X \times Y \to Y$ is a Z-map (that is, $\pi_Y(Z)$ is closed in Y for any zero-set Z of $X \times Y$) for any space Y.*

(3) *The projection $\pi_Y: X \times Y \to Y$ is a Z-map for any paracompact Hausdorff space Y.*

Proof. Since (2)→(3) is trivial and (3)→(1) is a direct consequence of Proposition 2.6, we prove only (1)→(2). Let Z be any zero-set of $X \times Y$ and put $Z = \{(x, y) \mid h(x, y) = 0\} \subset X \times Y$, where $h: X \times Y \to I$ is continuous. Now suppose there is a point y_0 of $Y - \pi_Y(Z)$ such that $(X \times V) \cap Z \neq \emptyset$ for any open nbd V of y_0. Since X is pseudocompact, we have

$$\inf_{x \in X} h(x, y_0) = a > 0,$$

and so for any point x of X there exists an open nbd $U_x \times V_x$ of (x, y_0) such that $h(z) > \frac{1}{2}a$ for any point z of $U_x \times V_x$. From this fact it easily follows that for any finite number of points $x(1), \ldots, x(n)$ of X there exists a zero-set $L(x(1), \ldots, x(n))$ of X such that

$$\bigcup_{i=1}^{n} U_{x(i)} \subset L(x(1), \ldots, x(n)) \neq X.$$

Since X is w-compact, this leads us a contradiction. Thus π_Y is a Z-map. This completes the proof.

If we replace "w-compact" by "pseudocompact" in Propositions 2.6 and 2.7, we have the following results, in which ω_0 is the first countable ordinal and $W(\omega_0 + 1)$ is the space of ordinals $\alpha \leq \omega_0$ with the usual interval topology.

Proposition 2.8. *Let U be a cozero-set of a space X such that $\bar{U}$ is not pseudocompact. Then there exists a continuous map $h: X \times W(\omega_0 + 1) \to I$ such that*

$$h(z) = 1 \quad \textit{for } z \in (X \times \omega_0) \cup ((X - U) \times W(\omega_0 + 1))$$

and

$$h^{-1}(0)\cap(U\times n)\neq\emptyset \quad \text{for } n<\omega_0.$$

Proof. Since $\bar{U}$ is not pseudocompact, there exists a decreasing sequence $\{Z_n\}$ of non-empty zero-sets of $\bar{U}$ with $\bigcap Z_n=\emptyset$. With no loss of generality, we may assume that for each n, Z_n contains a non-empty cozero-set G_n of $\bar{U}$. If we put $H_n=G_n\cap U$ for any n, then each H_n is a non-empty cozero-set of X. For any n, we take a non-empty zero-set L_n of X contained in H_n and consider a continuous map $h_n:X\times n\to I$ such that

$$h_n(z)=1 \quad \text{for } z\in(X-H_n)\times n,$$

$$h_n(z)=0 \quad \text{for } z\in L_n\times n.$$

Then the map $h:X\times W(\omega_0+1)\to I$ defined by

$$h(z)=h_n(z) \quad \text{for } z\in X\times n,$$

$$h(z)=1 \quad \text{for } z\in X\times\omega_0$$

is obviously continuous and satisfies the required properties. This completes the proof.

Proposition 2.9. *For a space X the following properties are equivalent.*

(1) *X is pseudocompact.*

(2) *The projection $\pi_Y:X\times Y\to Y$ is a Z-map for any k-space Y.*

(3) *The projection $\pi_Y:X\times Y\to Y$ is a Z-map for any compact Hausdorff space Y.*

(4) *The projection $\pi_Y:X\times Y\to Y$ is a Z-map for $Y=W(\omega_0+1)$.*

Proof. Since (2)$\to$(3) and (3)$\to$(4) are obvious and (4)$\to$(1) is a direct consequence of Proposition 2.8, we prove only (1)$\to$(2). For this purpose we may assume that Y is a compact Hausdorff space, sine Y is a k-space. Let $Z=\{(x,y)\mid h(x,y)=0\}$ be a zero-set of $X\times Y$, where $h:X\times Y\to I$ is continuous. By Theorem 1.1 we have $\tau(X\times Y)=\tau(X)\times Y$, and so there exists a continuous map $g:\tau(X)\times Y\to I$ such that $g\circ(\Phi_X\times i_Y)=h$, where $i_Y:Y\to Y$ is the identity map. Let us put $Z'=\{(x',y)\mid g(x',y)=0\}$ in $\tau(X)\times Y$. Since $(\Phi_X\times i_Y)^{-1}(Z')=Z$, we have $\pi_Y(Z)=\pi'_Y(Z')$, where $\pi'_Y:\tau(X)\times Y\to Y$ is the projection. Hence it suffices to show that $\pi'_Y(Z')$ is closed in Y. To see this, notice that $\tau(X)\times Y$ is pseudocompact as well as $\tau(X)$. Then, by a theorem of I. Glicksberg [2], $\tau(X)\times Y$ is C^*-embedded in $\beta(\tau(X))\times Y$, from which it easily follows that $\pi'_Y(Z')$ is closed in Y. Thus the proof is completed.

3. Proof of Theorem 1.5

Before proving Theorem 1.5, we state a lemma which is useful for the proof of Theorem 1.7 as well as Theorem 1.5.

Lemma 3.1. *There exists a regular k-space X containing two different points a and b such that $g(a)=g(b)$ for any continuous map $g: X\to R$.*

Proof. Let S, P, Q, S_n and φ_n be the same as in the proof of Proposition 2.3 with the exception that n is any integer. In the topological sum T of $\{S_n \mid n=0, \pm 1, \dots\}$, we identify a point $\varphi_{2m-1}(p)$ with $\varphi_{2m}(p)$ for $p\in P$ and a point $\varphi_{2m}(q)$ with $\varphi_{2m+1}(q)$ for $q\in Q$. By this identification we have a quotient space Y, which is locally compact Hausdorff. Let X be a space obtained by adding different new points a and b to Y and introducing the topology in X as follows: the bases at a and b are given by the totality of the sets

$$\{a\}\cup\Big(Y-\bigcup_{j\le n}\varphi_j(S)\Big) \quad\text{and}\quad \{b\}\cup\Big(Y-\bigcup_{j\ge -n}\varphi_j(S)\Big),\quad n\in N,$$

respectively. Then X is a regular space such that $g(a)=g(b)$ for any continuous map $g: X\to R$. Furthermore X is a k-space, because Y is locally compact Hausdorff and both a and b have the countable bases. This completes the proof.

Proof of Theorem 1.5. (1)$\to$(2). We first notice that $\tau(X)$ is locally compact. Indeed, let $x_0\in X$ and $t_0=\Phi_X(x_0)$. By our assumption there is a cozero-set nbd U of x_0 such that $\bar U$ is w-compact. Since $\Phi_X(\bar U)$ is compact by Proposition 2.5 and Corollary 2.2 and $\Phi_x(U)\subset\Phi_X(\bar U)\subset\overline{\Phi_X(U)}$, we have $\Phi_X(\bar U)=\overline{\Phi_X(U)}$, and hence $\overline{\Phi_X(U)}$ is a compact nbd of t_0.

Let H be a cozero-set of $X\times Y$ and let $(x,y)\in H$. Then there exists a cozero-set nbd U of x such that $\bar U\subset H(y)=\{x\in X\mid (x,y)\in H\}$ and $\bar U$ is w-compact; this easily follows from Proposition 2.4. Since the projection $\pi_Y:\bar U\times Y\to Y$ is a Z-map by Proposition 2.7, $V=Y-\pi_Y(\bar U\times Y-H)$ is an open nbd of y, and hence we have $U\times V\subset H$. From this fact it follows that, if $h: X\times Y\to I$ is continuous and if we put

$$h'(x',y)=h(x,y)\quad\text{for } (x',y)\in\tau(X)\times Y,$$

where x is an arbitrary point of $\Phi_X^{-1}(x')$, then $h':\tau(X)\times Y\to I$ is a (well-defined) continuous map. In fact, for any $\varepsilon>0$ and for any point (x_0,y_0) of $X\times Y$, there exists an open nbd $U\times V$ of (x_0,y_0) such that

$$U\times V\subset\{(x,y)\mid h(x_0,y_0)-\varepsilon<h(x,y)<h(x_0,y_0)+\varepsilon\}$$

and that U is a cozero-set of X. Since $\Phi_X(U)$ is a cozero-set of $\tau(X)$ and

$$h'(x_0',y_0)-\varepsilon<h'(x',y)<h'(x_0',y_0)+\varepsilon$$

for any $(x',y)\in\Phi_X(U)\times V$, where $x_0'=\Phi_X(x_0)$, h' is a continuous map.

Now let $G=\{(x,y)\mid g(x,y)>0\}$ be a cozero-set of $X\times Y$, where $g: X\times Y\to I$ is continuous, and let $(x_0,y_0)\in G$. If we define a map $g':\tau(X)\times Y\to I$ as above, it is continuous and so $G'=\{(x',y)\mid g'(x',y)>0\}$ is a cozero-set of $\tau(X)\times Y$. Let $x_0'=\Phi_X(x_0)$. Then, since $\tau(X)$ is locally compact, by Theorems 1.1 and 1.2 there exists a rectangular cozero-set $P\times Q$ of $\tau(X)\times Y$ such that

$$(x_0', y_0)\in P\times Q\subset G',$$

and hence

$$(x_0, y_0)\in \Phi_X^{-1}(P)\times Q\subset G,$$

where $\Phi_X^{-1}(P)\times Q$ is a rectangular cozero-set of $X\times Y$. Therefore by Theorem 1.2 we have $\tau(X\times Y)=\tau(X)\times\tau(Y)$.

(3)→(1). Suppose that there exists a point x_0 of X such that the closure of any cozero-set nbd of x_0 is not w-compact. If we denote by $\{W_\alpha \mid \alpha\in\Omega\}$ the totality of cozero-set nbds of x_0, then by Proposition 2.6, for any $\alpha\in\Omega$ there exists a space Y_α with only one non-isolated point $y_\alpha\in Y_\alpha$ and a continuous map $h_\alpha: X\times Y_\alpha\to I$ such that $h_\alpha(z)=1$ for $z\in X\times y_\alpha$ and $h_\alpha^{-1}(0)\cap(W_\alpha\times y)\neq\emptyset$ for $y\in Y_\alpha-y_\alpha$. Let S be a regular Hausdorff space constructed in Lemma 3.1, and let S_α be the copy of S for each $\alpha\in\Omega$. Then there exists a homeomorphism φ_α of S onto S_α. In the topological sum $S_\alpha\cup Y_\alpha$ we identify a point $b_\alpha=\varphi_\alpha(b)$ with y_α. By this identification we have a quotient space Z_α which is regular. We next define a space Y as a quotient space of the topological sum $\bigcup Z_\alpha$ obtained by identifying every point $a_\alpha=\varphi_\alpha(a)\in S_\alpha$. Clearly Y is regular Hausdorff. Let $q:\bigcup Z_\alpha\to Y$ be the quotient map, and put $\xi=q(a_\alpha)$ for each $\alpha\in\Omega$. Then it is shown that there exists a cozero-set G of $X\times Y$ containing (x_0,ξ), for which there is no rectangular cozero-set $U\times V$ such that $(x_0,\xi)\in U\times V\subset G$. To see this, we consider a map $g: X\times Y\to I$ defined by

$$g(z)=1 \qquad \text{for } z\in X\times(Y-\bigcup(Y_\alpha-y_\alpha)),$$

$$g(z)=h_\alpha(z) \quad \text{for } z\in X\times y,\ y\in Y_\alpha-y_\alpha.$$

Clearly it is a continuous map and so $G=\{(x,y)\mid g(x,y)>0\}$ is a cozero-set of $X\times Y$ containing (x_0,ξ). Furthermore we have $(W_\alpha\times V)\cap g^{-1}(0)\neq\emptyset$ for any rectangular cozero-set nbd $W_\alpha\times V$ of (x_0,ξ). In fact, for any cozero-set nbd V of ξ, we have $V\cap(Y_\alpha-y_\alpha)\neq\emptyset$ for each α, and hence there is no rectangular cozero-set $W_\alpha\times V$ such that $(x_0,\xi)\in W_\alpha\times V\subset G$, which implies that $\tau(X\times Y)\neq\tau(X)\times\tau(Y)$. This is a contradiction. Thus (3) implies (1).

Since (2)→(3) is obvious, we complete the proof.

4. Proof of Theorem 1.7

(1)→(2). Let X be a space satisfying (1) and Y an arbitrary k-space. If $G=\{(x,y)\mid g(x,y)>0\}$ is a cozero-set of $X\times Y$ with $(x_0,y_0)\in G$, where $g: X\times Y\to I$ is a continuous map, then there exists a cozero-set nbd U of x_0 such that $\bar U\subset G(y_0)$ and $\bar U$ is pseudocompact. By Proposition 2.9, $\pi_Y(\bar U\times Y-G)$ is closed in Y so that $V=Y-\pi_Y(\bar U\times Y-G)$ is open in Y, and

$$(x_0, y_0)\in U\times V\subset G.$$

From this fact it follows that, if we define a map $g':\tau(X)\times Y\to I$ by

$$g'(x',y)=g(x,y)\quad \text{for } (x',y)\in\tau(X)\times Y,$$

where x is an arbitrary point of $\Phi_X^{-1}(x')$, then it is a (well-defined) continuous map. Hence $G'=\{(x',y)\mid g'(x',y)>0\}$ is a cozero-set of $\tau(X)\times Y$ with $(x'_0,y_0)\in G'$, where $x'_0=\Phi_X(x_0)$. Therefore there exists a cozero-set nbd W of x'_0 such that $\bar{W}\subset G'(y_0)$ and $\bar{W}$ is pseudocompact. Indeed, let P be a cozero-set nbd of x'_0 such that $\bar{P}\subset G'(y_0)$. Then by (1) there exists a cozero-set nbd U_1 of x_0 such that $\bar{U}_1\subset\Phi_X^{-1}(P)$ and $\bar{U}_1$ is pseudocompact. Since $\Phi_X(U_1)\subset\Phi_X(\bar{U}_1)\subset\overline{\Phi_X(U_1)}$ and $\Phi_X(\bar{U}_1)$ is pseudocompact, $\overline{\Phi_X(U_1)}$ is also pseudocompact. Hence if we put $W=\Phi_X(U_1)$, it satisfies the required properties.

Now let K be any compact set of Y. Since $\bar{W}\times K$ is pseudocompact, it is C^*-embedded in $\beta(\bar{W})\times K$ by I. Glicksberg's theorem [2]. Hence from the fact that $\beta(\bar{W})\times Y$ is a k-space, it follows that $\bar{W}\times Y$ is C^*-embedded in $\beta(\bar{W})\times Y$. Let $g'\mid\bar{W}\times Y$ be the restriction of $g':\tau(X)\times Y\to I$ to $\bar{W}\times Y$ and $g'_\beta:\beta(\bar{W})\times Y\to I$ a continuous extension of $g'\mid\bar{W}\times Y$ over $\beta(\bar{W})\times Y$. If we put

$$G'_\beta=\{(x,y)\in\beta(\bar{W})\times Y\mid g'_\beta(x,y)>0\},$$

then $G'\cap(\bar{W}\times Y)=G'_\beta\cap(\bar{W}\times Y)$, and $\beta(\bar{W})\times y_0\subset G'_\beta$ by the pseudocompactness of $\bar{W}$. Furthermore a map $f:Y\to I$ defined by

$$f(y)=\inf\{g'_\beta(x,y)\mid x\in\beta(\bar{W})\}$$

is continuous by the compactness of $\beta(\bar{W})$. Hence if we put

$$Q=\{y\in Y\mid f(y)>0\},$$

we have $\beta(\bar{W})\times Q\subset G'_\beta$. Since this implies that $W\times Q\subset G'$, it follows that

$$(x_0,y_0)\in\Phi_X^{-1}(W)\times Q\subset G.$$

Hence by Theorem 1.2 we have $\tau(X\times Y)=\tau(X)\times\tau(Y)$.

(3)$\to$(1). Suppose that there exists a point x_0 of X and a cozero-set nbd U of x_0 such that for any cozero-set nbd W of x_0 with $W\subset U$, $\bar{W}$ is not pseudocompact. Let $\{W_\alpha\mid\alpha\in\Omega\}$ be the totality of cozero-set nbds of x_0 contained in U. Then by Proposition 2.8, for each $\alpha\in\Omega$ there exists a space $Y_\alpha=W(\omega_0+1)$ and a continuous map $h_\alpha:X\times Y_\alpha\to I$ such that

$$h_\alpha(z)=1\qquad\text{for } z\in X\times\omega_0,$$

$$h_\alpha^{-1}(0)\cap(W_\alpha\times n)\neq\emptyset\quad\text{for } n<\omega_0.$$

Let S be the regular k-space constructed in Lemma 3.1, and let S_α be the copy of S for each $\alpha\in\Omega$. Then there exists a homeomorphism $\varphi_\alpha:S\to S_\alpha$, and so we put $a_\alpha=\varphi_\alpha(a)$ and $b_\alpha=\varphi_\alpha(b)$. Let P_α be a space obtained from the topological sum $S_\alpha\cup Y_\alpha$ by identifying $b_\alpha\in S_\alpha$ and $\omega_0\in Y_\alpha$. We next define a space Y as a quotient space of the topological sum $\bigcup P_\alpha$ obtained by identifying every a_α. Since the topological sum $S_\alpha\cup Y_\alpha$ is a regular k-space, so is the quotient space P_α. Hence both the topological sum $\bigcup P_\alpha$ and its quotient space Y are also regular k-spaces. Let $q:\bigcup P_\alpha \to Y$ be a quotient map, and let us put $\xi = q(a_\alpha)$, $\alpha \in \Omega$. Then by the same way as in the proof of Theorem 1.5 we can construct a cozero-set G of $X \times Y$ with $(x_0, \xi) \in G$, for which there is no rectangular cozero-set $U \times V$ such that $(x_0, \xi) \in U \times V \subset G$. Therefore we have $\tau(X \times Y) \neq \tau(X) \times \tau(Y)$, which is a contradiction. This shows that (3) implies (1).

Since (2)→(3) is obvious, we complete the proof.

5. Proof of Theorem 1.8

As a first step we shall prove the following lemma.

Lemma 5.1. *Let $\mathcal{M} = \{P_\lambda \mid \lambda \in \Lambda\}$ be any maximal family of τ-open sets of the product $X = \prod\{X_i \mid i = 1, \ldots, n\}$ of spaces $X_1, \ldots, X_n$ with the finite intersection property. If $x_i \in \bigcap\{\mathrm{cl}_{X_i}\pi_i(P_\lambda) \mid \lambda \in \Lambda\}$ for $i = 1, \ldots, n$, then*

$$(x_1, \ldots, x_n) \in \bigcap\{\mathrm{cl}_X P_\lambda \mid \lambda \in \Lambda\},$$

where $\pi_i : X \to X_i$ is the projection.

Therefore, if X_i, $i = 1, \ldots, n$, are w-compact, then $\prod\{X_i \mid i = 1, \ldots, n\}$ is also w-compact.

Proof. To prove the first part of the lemma by induction, assume that it is valid for $n = k-1$, where $2 \leqslant k$. Notice that, in case $n = 1$, it is obviously valid. Now let $\mathcal{M} = \{P_\lambda \mid \lambda \in \Lambda\}$ be any maximal family of τ-open sets of $X = \prod\{X_i \mid i = 1, \ldots, k\}$ with the finite intersection property and let $x_i \in \bigcap\{\mathrm{cl}_{X_i}\pi_i(P_\lambda) \mid \lambda \in \Lambda\}$ for $i = 1, \ldots, k$. If we denote by π the projection from X to $X' = \prod\{X_i \mid i = 1, \ldots, k-1\}$, then $\mathcal{M}' = \{\pi(P_\lambda) \mid \lambda \in \Lambda\}$ is a maximal family of τ-open sets of X' with the finite intersection property and $x_i \in \bigcap\{\mathrm{cl}_{X_i}\pi'_i(\pi(P_\lambda)) \mid \lambda \in \Lambda\}$ for $i = 1, \ldots, k-1$, where π'_i is the projection from X' to X_i. So by our assumption we have

$$(x_1, \ldots, x_{k-1}) \in \bigcap\{\mathrm{cl}_{X'}\pi(P_\lambda) \mid \lambda \in \Lambda\}.$$

To show that

$$x = (x_1, \ldots, x_k) \in \bigcap\{\mathrm{cl}_X P_\lambda \mid \lambda \in \Lambda\},$$

we take any open nbd $U_1 \times \cdots \times U_k$ of x. As is easily seen, for any $\lambda \in \Lambda$ the set $\pi_k((U_1 \times \cdots \times U_{k-1} \times X_k) \cap P_\lambda)$ is a τ-open set of X_k belonging to a maximal family $\mathcal{M}_k = \{\pi_k(P_\lambda) \mid \lambda \in \Lambda\}$ of τ-open sets of X_k with the finite intersection property. Hence we have

$$U_k \cap \pi_k((U_1 \times \cdots \times U_{k-1} \times X_k) \cap P_\lambda) \neq \emptyset$$

for any $\lambda \in \Lambda$, which implies that

$$(U_1 \times \cdots \times U_k) \cap P_\lambda \neq \emptyset \quad \text{for } \lambda \in \Lambda.$$

Consequently we have $(x_1, \dots, x_k) \in \bigcap\{\mathrm{cl}_X P_\lambda \mid \lambda \in \Lambda\}$. Thus the first part of the lemma is valid.

To show the second part of the lemma, let X_i, $i = 1, \dots, n$, be w-compact spaces and $\mathscr{A}$ the family of τ-open sets of $X = \prod\{X_i \mid i = 1, \dots, n\}$ with the finite intersection property. Then by Zorn's lemma there exists a maximal family $\mathscr{M} = \{P_\lambda \mid \lambda \in \Lambda\}$ of τ-open sets of X with the finite intersection property containing $\mathscr{A}$. Since for each i $\{\pi_i(P_\lambda) \mid \lambda \in \Lambda\}$ is a maximal family of τ-open sets of a w-compact space X_i with the finite intersection property, there exists a point x_i of X_i such that

$$x_i \in \bigcap\{\mathrm{cl}_{X_i} \pi_i(P_\lambda) \mid \lambda \in \Lambda\}.$$

Hence by the first part of the lemma, we have

$$(x_1, \dots, x_n) \in \bigcap\{\mathrm{cl}_X P_\lambda \mid \lambda \in \Lambda\}.$$

Therefore it follows that X is w-compact. This completes the proof.

Proof of Theorem 1.8. (1) Let $\{X_\alpha \mid \alpha \in \Omega\}$ be a family of w-compact spaces, and let $\mathscr{M} = \{P_\lambda \mid \lambda \in \Lambda\}$ be any maximal family of τ-open sets of $\prod X_\alpha$ with the finite intersection property. Since each X_α is w-compact, there exists a point $x_\alpha \in X_\alpha$ such that

$$x_\alpha \in \bigcap\{\mathrm{cl}_{X_\alpha} \pi_\alpha(P_\lambda) \mid \lambda \in \Lambda\},$$

where π_α is the projection from $\prod X_\alpha$ to X_α. Let us put $x_0 = (x_\alpha) \in \prod X_\alpha$. To see that for any open nbd $\bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_i)$ of x_0,

$$\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_i) \cap P_\lambda \neq \emptyset \quad \text{for } \lambda \in \Lambda,$$

it suffices to show that

$$(U_1 \times \cdots \times U_n) \cap \pi(P_\alpha) \neq \emptyset \quad \text{for } \lambda \in \Lambda,$$

where π is the projection from $\prod X_\alpha$ to $\prod\{X_{\alpha_i} \mid i = 1, \dots, n\}$. However this is valid by Lemma 5.1, since $\{\pi(P_\lambda) \mid \lambda \in \Lambda\}$ is a maximal family of τ-open sets of $\prod\{X_{\alpha i} \mid i = 1, \dots, n\}$ with the finite intersection property. Thus (1) holds.

(2) Let $\{X_\alpha \mid \alpha \in \Omega\}$ be a family of w-compact spaces. Then by (1) the product $X = \prod X_\alpha$ is w-compact. Hence by Proposition 2.1, $\tau(X)$ is compact as well as any $\tau(X_\alpha)$. Denoting by Φ_α for brevity the natural map of a space X_α to $\tau(X_\alpha)$, there exists a continuous map $f : \tau(X) \to \prod \tau(X_\alpha)$ such that $\prod \Phi_\alpha = f \circ \Phi_X$, where $\prod \Phi_\alpha : X \to \prod \tau(X_\alpha)$ is the product map. Obviously f is surjective. Hence, to show that f is homeomorphic, it suffices to prove that f is one-to-one.

Let I^X be the set of all continuous maps $\varphi : X \to I$, and let $x = (x_\alpha)$ and $y = (y_\alpha)$ be any two points of X. If $\varphi(x) = \varphi(y)$ for any $\varphi \in I^X$, then for any $\alpha \in \Omega$

$$\varphi_\alpha(x_\alpha) = \varphi_\alpha(y_\alpha) \quad \text{for any } \varphi_\alpha \in I^{X_\alpha}.$$

We next prove that the converse is true. To see this, assume the contrary. Then there exists a $\varphi \in I^X$ such that

$$|\varphi(x)-\varphi(y)|>\varepsilon \quad \text{for some } \varepsilon>0.$$

We take an open nbd $\bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_i)$ of $x=(x_\alpha)$ such that

$$|\varphi(x)-\varphi(x')|<\varepsilon \quad \text{for } x' \in \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_i),$$

where π_α is the projection of X to X_α, and define a point $y'=(y'_\alpha)$ as follows:

$$y'_\alpha = y_\alpha \quad \text{for } \alpha \neq \alpha_1, \ldots, \alpha_n,$$
$$y'_{\alpha_i} = x_{\alpha_i} \quad \text{for } i=1, \ldots, n.$$

Since $y' \in \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_i)$, it holds that $|\varphi(y')-\varphi(x)|<\varepsilon$. On the other hand, from the assumption that $\varphi_\alpha(x_\alpha)=\varphi_\alpha(y_\alpha)$ for any $\varphi_\alpha \in I^{X_\alpha}$, it follows that $\varphi(y)=\varphi(y')$. To see this, let us put $y_0=y$ and define $y_k \in X$, $k=1, \ldots, n$, as follows:

$$(y_k)_{\alpha_i} = x_{\alpha_i}, \quad \text{for } i=1, \ldots, k,$$
$$(y_k)_\alpha = y_\alpha \quad \text{for } \alpha \neq \alpha_1, \ldots, \alpha_k,$$

where $(y_k)_\alpha$ denotes the α-coordinate of y_k. Then $y_n=y'$ and $\operatorname{card}\{\alpha \mid (y_{k-1})_\alpha \neq (y_k)_\alpha\} \leq 1$ for $k=1, \ldots, n$. Let us further define a map $\varphi_{\alpha_k} : X_{\alpha_k} \to I$ as the restriction of φ to the subset

$$x_{\alpha_1} \times \cdots \times x_{\alpha_{k-1}} \times X_{\alpha_k} \times \prod \{y_\alpha \mid \alpha \neq \alpha_1, \ldots, \alpha_k\} \quad \text{of } X.$$

Then, since $\varphi_{\alpha_k}(y_{\alpha_k})=\varphi_{\alpha_k}(x_{\alpha_k})$, we have

$$\varphi(x_{\alpha_1} \times \cdots \times x_{\alpha_{k-1}} \times y_{\alpha_k} \times \prod \{y_\alpha \mid \alpha \neq \alpha_1, \ldots, \alpha_k\}) =$$
$$= \varphi(x_{\alpha_1} \times \cdots \times x_{\alpha_{k-1}} \times x_{\alpha_k} \times \prod \{y_\alpha \mid \alpha \neq \alpha_1, \ldots, \alpha_k\}),$$

which means that

$$\varphi(y_{k-1}) = \varphi(y_k) \quad \text{for } k=1, \ldots, n.$$

Therefore the equality $\varphi(y)=\varphi(y')$ holds. Hence it follows that $|\varphi(x)-\varphi(y)|<\varepsilon$, which is a contradiction. Consequently the converse is true, which shows that f is one-to-one. Thus the proof is completed.

Remark. In the proof of Theorem 1.8 (2), we use only the fact that every $\tau(X_\alpha)$ is compact as well as $\tau(\prod X_\alpha)$. If $\tau(\prod X_\alpha)$ is compact, then $\prod \tau(X_\alpha)$ is compact as the image of $\tau(\prod X_\alpha)$ under a continuous map, and hence every $\tau(X_\alpha)$ is also compact. Therefore Theorem 1.8(2) holds under the assumption that $\tau(\prod X_\alpha)$ is compact. However I do not know whether $\tau(\prod X_\alpha)$ is compact whenever every $\tau(X_\alpha)$ is compact.

6. Proof of Corollaries 1.9 and 1.10

Let $\{X_\alpha \mid \alpha \in \Omega\}$ be a family of w-compact spaces, and let $f:\prod X_\alpha \to I$ be a continuous map. Then there exists a continuous map $f_1:\prod \tau(X_\alpha) \to I$ such that $f = f_1 \circ \Phi_{\prod X_\alpha}$, since $\tau(\prod X_\alpha) = \prod \tau(X_\alpha)$ by Theorem 1.8. Hence it follows from Y. Mibu's theorem [3] that there exists a countable number of indices $\{\alpha(i) \mid i \in N\}$ and a continuous map $h:\prod \tau(X_{\alpha(i)}) \to I$ such that $f_1 = h \circ \pi_1$, where $\pi_1:\prod \tau(X_\alpha) \to \prod \tau(X_{\alpha(i)})$ is the projection. Let us put $g = h \circ \Phi_{\prod X_{\alpha(i)}}$. Then $g:\prod X_{\alpha(i)} \to I$ satisfies the required property. Indeed, by $\tau(\prod X_{\alpha(i)}) = \prod \tau(X_{\alpha(i)})$, we have

$$\begin{aligned} f &= f_1 \circ \Phi_{\prod X_\alpha} = h \circ \pi_1 \circ \Phi_{\prod X_\alpha} \\ &= h \circ \Phi_{\prod X_{\alpha(i)}} \circ \pi = g \circ \pi, \end{aligned}$$

where $\pi:\prod X_\alpha \to \prod X_{\alpha(i)}$ is the projection. Thus the proof of Corollary 1.9 is completed.

Corollary 1.10 is also proved by reducing it to the case of compact spaces and making use of R. Engelking's theorem [1]. This completes the proof.

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